# Lyapunov Coefficients for Degenerate Hopf Bifurcations

Jorge Sotomayor

*Instituto de Matemática e Estatística, Universidade de São Paulo*

*Rua do Matão 1010, Cidade Universitária*

*CEP 05.508-090, São Paulo, SP, Brazil*

e–mail:sotp@ime.usp.br

Luis Fernando Mello

*Instituto de Ciências Exatas, Universidade Federal de Itajubá*

*Avenida BPS 1303, Pinheirinho, CEP 37.500-903, Itajubá, MG, Brazil*

e–mail:lfmelo@unifei.edu.br

Denis de Carvalho Braga

*Instituto de Sistemas Elétricos e Energia, Universidade Federal de Itajubá*

*Avenida BPS 1303, Pinheirinho, CEP 37.500-903, Itajubá, MG, Brazil*

e–mail:braga_denis@yahoo.com.br

## Abstract

In this paper are studied the codimensions one, two, three and four Hopf bifurcations and the pertinent Lyapunov stability coefficients. Algebraic expressions obtained with computer assisted calculations are displayed.

**Key-words**: Lyapunov coefficients, degenerate Hopf bifurcation.
**MSC**: 70K50, 70K20.

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# 1 Lyapunov coefficients

The beginning of this section is a review of the method found in [3] and in [4] for the calculation of the first and second Lyapunov coefficients. The calculation of the third Lyapunov coefficient can be found in [6]. The calculation of the fourth Lyapunov coefficient has not been found by the authors in the current literature. The extensive calculations and the long expressions for these coefficients have been obtained with the software MATHEMATICA 5 [11] and the main computational steps have been posted in the site [10].

Consider the differential equations

$$\mathbf{x}' = f(\mathbf{x}, \mu), \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^m$ are respectively vectors representing phase variables and control parameters. Assume that $f$ is of class $C^\infty$ in $\mathbb{R}^n \times \mathbb{R}^m$. Suppose (1) has an equilibrium point $\mathbf{x} = \mathbf{x_0}$ at $\mu = \mu_0$ and, denoting the variable $\mathbf{x} - \mathbf{x_0}$ also by $\mathbf{x}$, write

$$F(\mathbf{x}) = f(\mathbf{x}, \mu_0) \tag{2}$$

as

$$F(\mathbf{x}) = A\mathbf{x} + \frac{1}{2} B(\mathbf{x}, \mathbf{x}) + \frac{1}{6} C(\mathbf{x}, \mathbf{x}, \mathbf{x}) + \frac{1}{24} D(\mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}) + \frac{1}{120} E(\mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}) +$$
$$\frac{1}{720} K(\mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}) + \frac{1}{5040} L(\mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}) + \frac{1}{40320} M(\mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}) \tag{3}$$
$$+ \frac{1}{362880} N(\mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}) + O(||\mathbf{x}||^{10}),$$

where $A = f_\mathbf{x}(0, \mu_0)$ and

$$B_i(\mathbf{x}, \mathbf{y}) = \sum_{j,k=1}^n \left. \frac{\partial^2 F_i(\xi)}{\partial \xi_j \, \partial \xi_k} \right|_{\xi=0} x_j \, y_k, \tag{4}$$

$$C_i(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \sum_{j,k,l=1}^n \left. \frac{\partial^3 F_i(\xi)}{\partial \xi_j \, \partial \xi_k \, \partial \xi_l} \right|_{\xi=0} x_j \, y_k \, z_l, \tag{5}$$

$$D_i(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u}) = \sum_{j,k,l,r=1}^{n} \frac{\partial^4 F_i(\xi)}{\partial \xi_j \, \partial \xi_k \, \partial \xi_l \, \partial \xi_r}\bigg|_{\xi=0} x_j \, y_k \, z_l \, u_r, \qquad (6)$$

$$E_i(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{v}) = \sum_{j,k,l,r,p=1}^{n} \frac{\partial^5 F_i(\xi)}{\partial \xi_j \, \partial \xi_k \, \partial \xi_l \, \partial \xi_r \, \partial \xi_p}\bigg|_{\xi=0} x_j \, y_k \, z_l \, u_r \, v_p, \qquad (7)$$

$$K_i(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{v}, \mathbf{w}) = \sum_{j,\ldots,q=1}^{n} \frac{\partial^6 F_i(\xi)}{\partial \xi_j \, \partial \xi_k \, \partial \xi_l \, \partial \xi_r \, \partial \xi_p \, \partial \xi_q}\bigg|_{\xi=0} x_j \, y_k \, z_l \, u_r \, v_p \, w_q, \qquad (8)$$

$$L_i(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{t}) = \sum_{j,\ldots,h=1}^{n} \frac{\partial^7 F_i(\xi)}{\partial \xi_j \partial \xi_k \partial \xi_l \partial \xi_r \partial \xi_p \partial \xi_q \partial \xi_h}\bigg|_{\xi=0} x_j \, y_k \, z_l \, u_r \, v_p \, w_q \, t_h, \qquad (9)$$

$$M_i(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{t}, \mathbf{r}) = \sum_{j,\ldots,a=1}^{n} \frac{\partial^8 F_i(\xi)}{\partial \xi_j \ldots \partial \xi_h \partial \xi_a}\bigg|_{\xi=0} x_j \, y_k \, z_l \, u_r \, v_p \, w_q \, t_h \, r_a, \qquad (10)$$

$$N_i(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{t}, \mathbf{r}, \mathbf{s}) = \sum_{j,\ldots,b=1}^{n} \frac{\partial^9 F_i(\xi)}{\partial \xi_j \ldots \partial \xi_b}\bigg|_{\xi=0} x_j \, y_k \, z_l \, u_r \, v_p \, w_q \, t_h \, r_a \, s_b, \qquad (11)$$

for $i = 1, \ldots, n$.

Suppose $(\mathbf{x_0}, \mu_0)$ is an equilibrium point of (1) where the Jacobian matrix $A$ has a pair of purely imaginary eigenvalues $\lambda_{2,3} = \pm i\omega_0$, $\omega_0 > 0$, and admits no other eigenvalue with zero real part. Let $T^c$ be the generalized eigenspace of $A$ corresponding to $\lambda_{2,3}$. By this is meant that it is the largest subspace invariant by $A$ on which the eigenvalues are $\lambda_{2,3}$.

Let $p, q \in \mathbb{C}^n$ be vectors such that

$$Aq = i\omega_0 \, q, \quad A^\top p = -i\omega_0 \, p, \quad \langle p, q \rangle = \sum_{i=1}^{n} \bar{p}_i \, q_i \; = 1, \qquad (12)$$

where $A^\top$ is the transposed of the matrix $A$. Any vector $y \in T^c$ can be represented as $y = wq + \bar{w}\bar{q}$, where $w = \langle p, y \rangle \in \mathbb{C}$. The two dimensional center manifold can be parameterized by $w, \bar{w}$, by means of an immersion of the form $\mathbf{x} = H(w, \bar{w})$, where $H : \mathbb{C}^2 \to \mathbb{R}^n$ has a Taylor expansion of the form

$$H(w, \bar{w}) = wq + \bar{w}\bar{q} + \sum_{2 \leq j+k \leq 9} \frac{1}{j!k!} \, h_{jk} w^j \bar{w}^k + O(|w|^{10}), \qquad (13)$$

with $h_{jk} \in \mathbb{C}^n$ and $h_{jk} = \bar{h}_{kj}$. Substituting this expression into (1) we obtain the following differential equation

$$H_w w' + H_{\bar{w}} \bar{w}' = F(H(w, \bar{w})), \tag{14}$$

where $F$ is given by (2).

The complex vectors $h_{ij}$ are obtained solving the system of linear equations defined by the coefficients of (14), taking into account the coefficients of $F$, so that system (14), on the chart $w$ for a central manifold, writes as follows

$$w' = i\omega_0 w + \frac{1}{2} G_{21} w |w|^2 + \frac{1}{12} G_{32} w |w|^4 + \frac{1}{144} G_{43} w |w|^6 + \frac{1}{2880} G_{54} w |w|^8 + O(|w|^{10}),$$

with $G_{jk} \in \mathbb{C}$.

Solving for the vectors $h_{ij}$ the system of linear equations defined by the coefficients of the quadratic terms of (14), taking into account the coefficients of $F$ in the expressions (3) and (4), one has

$$h_{11} = -A^{-1} B(q, \bar{q}), \tag{15}$$

$$h_{20} = (2i\omega_0 I_n - A)^{-1} B(q, q), \tag{16}$$

where $I_n$ is the unit $n \times n$ matrix. Pursuing the calculation to cubic terms, from the coefficients of the terms $w^3$ in (14) follows that

$$h_{30} = (3i\omega_0 I_n - A)^{-1} \left[ 3B(q, h_{20}) + C(q, q, q) \right]. \tag{17}$$

From the coefficients of the terms $w^2 \bar{w}$ in (14) one obtains a singular system for $h_{21}$

$$(i\omega_0 I_n - A) h_{21} = C(q, q, \bar{q}) + B(\bar{q}, h_{20}) + 2B(q, h_{11}) - G_{21} q, \tag{18}$$

which has a solution if and only if

$$\langle p, C(q, q, \bar{q}) + B(\bar{q}, h_{20}) + 2B(q, h_{11}) - G_{21} q \rangle = 0.$$

The *first Lyapunov coefficient* $l_1$ is defined by

$$l_1 = \frac{1}{2} \operatorname{Re} G_{21}, \tag{19}$$

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where

$$G_{21} = \langle p, \mathcal{H}_{21} \rangle, \text{ and } \mathcal{H}_{21} = C(q, q, \bar{q}) + B(\bar{q}, h_{20}) + 2B(q, h_{11}).$$

The complex vector $h_{21}$ can be found by solving the nonsingular $(n+1)$-dimensional system

$$\begin{pmatrix} i\omega_0 I_n - A & q \\ \bar{p} & 0 \end{pmatrix} \begin{pmatrix} h_{21} \\ s \end{pmatrix} = \begin{pmatrix} \mathcal{H}_{21} - G_{21}q \\ 0 \end{pmatrix}, \qquad (20)$$

with the condition $\langle p, h_{21} \rangle = 0$. The procedure above can be adapted in connection with the determination of $h_{32}$ and $h_{43}$.

For the sake of completeness, in Remark 1.1 we prove that the system (20) is nonsingular and that if $(v, s)$ is a solution of (20) with the condition $\langle p, v \rangle = 0$ then $v$ is a solution of (18). See Remark 3.1 of [6].

**Remark 1.1** *Write $\mathbb{R}^n = T^c \oplus T^{su}$, where $T^c$ and $T^{su}$ are invariant by $A$. It can be proved that $y \in T^{su}$ if and only if $\langle p, y \rangle = 0$. Define*

$$a = C(q, q, \bar{q}) + B(\bar{q}, h_{20}) + 2B(q, h_{11}) - G_{21}q.$$

*Let $(v, s)$ be a solution of the homogeneous equation obtained from (20). Equivalently*

$$(i\omega_0 I_n - A)v + sq = 0, \quad \langle p, v \rangle = 0. \qquad (21)$$

*From the second equation of (21), it follows that $v \in T^{su}$, and thus $(i\omega_0 I_n - A)v \in T^{su}$. Therefore $\langle p, (i\omega_0 I_n - A)v \rangle = 0$. Taking the inner product of $p$ with the first equation of (21) one has $\langle p, (i\omega_0 I_n - A)v + sq \rangle = 0$, which can be written as $\langle p, (i\omega_0 I_n - A)v \rangle + s\langle p, q \rangle = 0$. Since $\langle p, q \rangle = 1$ and $\langle p, (i\omega_0 I_n - A)v \rangle = 0$ it follows that $s = 0$. Substituting $s = 0$ into the first equation of (21) one has $(i\omega_0 I_n - A)v = 0$. This implies that*

$$v = \alpha q, \ \alpha \in \mathbb{C}. \qquad (22)$$

But $0 = \langle p, v \rangle = \langle p, \alpha q \rangle = \alpha \langle p, q \rangle = \alpha$. *Substituting $\alpha = 0$ into (22) it follows that $v = 0$. Therefore $(v, s) = (0, 0)$.*

*Let $(v, s)$ be a solution of (20). Equivalently*

$$(i\omega_0 I_n - A)v + sq = a, \quad \langle p, v \rangle = 0. \tag{23}$$

*From the second equation of (23), it follows that $v \in T^{su}$ and thus $(i\omega_0 I_n - A)v \in T^{su}$. Therefore $\langle p, (i\omega_0 I_n - A)v \rangle = 0$. Taking the inner product of $p$ with the first equation of (23) one has $\langle p, (i\omega_0 I_n - A)v + sq \rangle = \langle p, a \rangle$, which can be written as*

$$\langle p, (i\omega_0 I_n - A)v \rangle + s\langle p, q \rangle = \langle p, a \rangle.$$

*As $\langle p, a \rangle = 0$, $\langle p, q \rangle = 1$ and $\langle p, (i\omega_0 I_n - A)v \rangle = 0$ it follows that $s = 0$. Substituting $s = 0$ into the first equation of (23) results $(i\omega_0 I_n - A)v = a$. Therefore $v$ is a solution of (18).*

From the coefficients of the terms $w^4$, $w^3 \bar{w}$ and $w^2 \bar{w}^2$ in (14), one has respectively

$$h_{40} = (4i\omega_0 I_n - A)^{-1}[3B(h_{20}, h_{20}) + 4B(q, h_{30}) + 6C(q, q, h_{20}) + D(q, q, q, q)], \tag{24}$$

$$\begin{aligned} h_{31} = (2i\omega_0 I_n - A)^{-1}[&3B(q, h_{21}) + B(\bar{q}, h_{30}) + 3B(h_{20}, h_{11}) \\ &+ 3C(q, q, h_{11}) + 3C(q, \bar{q}, h_{20}) + D(q, q, q, \bar{q}) - 3G_{21}h_{20}], \end{aligned} \tag{25}$$

$$\begin{aligned} h_{22} = -A^{-1}[&D(q, q, \bar{q}, \bar{q}) + 4C(q, \bar{q}, h_{11}) + C(\bar{q}, \bar{q}, h_{20}) + C(q, q, \bar{h}_{20}) \\ &+ 2B(h_{11}, h_{11}) + 2B(q, \bar{h}_{21}) + 2B(\bar{q}, h_{21}) + B(\bar{h}_{20}, h_{20})], \end{aligned} \tag{26}$$

where the term $-2h_{11}(G_{21} + \bar{G}_{21})$ has been omitted in the last equation, since $G_{21} + \bar{G}_{21} = 0$ as $l_1 = 0$.

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Defining $\mathcal{H}_{32}$ as

$$
\begin{aligned}
\mathcal{H}_{32} = {} & 6B(h_{11}, h_{21}) + B(\bar{h}_{20}, h_{30}) + 3B(\bar{h}_{21}, h_{20}) + 3B(q, h_{22}) \\
& +2B(\bar{q}, h_{31}) + 6C(q, h_{11}, h_{11}) + 3C(q, \bar{h}_{20}, h_{20}) + 3C(q, q, \bar{h}_{21}) \\
& +6C(q, \bar{q}, h_{21}) + 6C(\bar{q}, h_{20}, h_{11}) + C(\bar{q}, \bar{q}, h_{30}) + D(q, q, q, \bar{h}_{20}) \\
& +6D(q, q, \bar{q}, h_{11}) + 3D(q, \bar{q}, \bar{q}, h_{20}) + E(q, q, q, \bar{q}, \bar{q}) \\
& \hspace{5cm} -6G_{21}h_{21} - 3\bar{G}_{21}h_{21},
\end{aligned}
$$

and from the coefficients of the terms $w^3\bar{w}^2$ in (14), one has a singular system for $h_{32}$

$$
(i\omega_0 I_n - A)h_{32} = \mathcal{H}_{32} - G_{32}q, \tag{27}
$$

which has solution if and only if

$$
\langle p, \mathcal{H}_{32} - G_{32}q \rangle = 0. \tag{28}
$$

where the terms $-6G_{21}h_{21} - 3\bar{G}_{21}h_{21}$ in the last line of (27) actually does not enter in last equation, since $\langle p, h_{21} \rangle = 0$.

The *second Lyapunov coefficient* is defined by

$$
l_2 = \frac{1}{12} \operatorname{Re} G_{32}, \tag{29}
$$

where, from (28), $G_{32} = \langle p, \mathcal{H}_{32} \rangle$.

The complex vector $h_{32}$ can be found solving the nonsingular $(n+1)$-dimensional system

$$
\begin{pmatrix} i\omega_0 I_n - A & q \\ \bar{p} & 0 \end{pmatrix} \begin{pmatrix} h_{32} \\ s \end{pmatrix} = \begin{pmatrix} \mathcal{H}_{32} - G_{32}q \\ 0 \end{pmatrix}, \tag{30}
$$

with the condition $\langle p, h_{32} \rangle = 0$.

From the coefficients of the terms $w^4\bar{w}$, $w^4\bar{w}^2$ and $w^3\bar{w}^3$ in (14), one has respectively

$$
h_{41} = (3i\omega_0 I_n - A)^{-1}[4B(h_{11}, h_{30}) + 6B(h_{20}, h_{21}) + 4B(q, h_{31})
$$

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$$+B(\bar{q}, h_{40}) + 12C(q, h_{11}, h_{20}) + 6C(q, q, h_{21}) + 4C(q, \bar{q}, h_{30}) \quad (31)$$
$$+3C(\bar{q}, h_{20}, h_{20}) + 4D(q, q, q, h_{11}) + 6D(q, q, \bar{q}, h_{20})$$
$$+E(q, q, q, q, \bar{q}) - 6G_{21}h_{30}],$$

$$h_{42} = (2i\omega_0 I_n - A)^{-1}[8B(h_{11}, h_{31}) + 6B(h_{20}, h_{22}) + B(\bar{h}_{20}, h_{40})$$
$$+6B(h_{21}, h_{21}) + 4B(\bar{h}_{21}, h_{30}) + 4B(q, h_{32}) + 2B(\bar{q}, h_{41})$$
$$+12C(h_{11}, h_{11}, h_{20}) + 3C(h_{20}, h_{20}, \bar{h}_{20}) + 24C(q, h_{11}, h_{21})$$
$$+12C(q, h_{20}, \bar{h}_{21}) + 4C(q, \bar{h}_{20}, h_{30}) + 6C(q, q, h_{22}) + 8C(q, \bar{q}, h_{31})$$
$$+8C(\bar{q}, h_{11}, h_{30}) + 12C(\bar{q}, h_{20}, h_{21}) + C(\bar{q}, \bar{q}, h_{40}) \quad (32)$$
$$+12D(q, q, h_{11}, h_{11}) + 6D(q, q, h_{20}, \bar{h}_{20}) + 4D(q, q, q, \bar{h}_{21})$$
$$+12D(q, q, \bar{q}, h_{21}) + 24D(q, \bar{q}, h_{11}, h_{20}) + 4D(q, \bar{q}, \bar{q}, h_{30})$$
$$+3D(\bar{q}, \bar{q}, h_{20}, h_{20}) + E(q, q, q, q, \bar{h}_{20}) + 8E(q, q, q, \bar{q}, h_{11})$$
$$+6E(q, q, \bar{q}, \bar{q}, h_{20}) + K(q, q, q, q, \bar{q}, \bar{q})$$
$$-4(G_{32}h_{20} + 3G_{21}h_{31} + \bar{G}_{21}h_{31})],$$

$$h_{33} = -A^{-1}[9B(h_{11}, h_{22}) + 3B(h_{20}, \bar{h}_{31}) + 3B(\bar{h}_{20}, h_{31}) + 9B(h_{21}, \bar{h}_{21})$$
$$+B(\bar{h}_{30}, h_{30}) + 3B(q, \bar{h}_{32}) + 3B(\bar{q}, h_{32}) + 6C(h_{11}, h_{11}, h_{11})$$
$$+9C(h_{11}, \bar{h}_{20}, h_{20}) + 18C(q, h_{11}, \bar{h}_{21}) + 3C(q, h_{20}, \bar{h}_{30})$$
$$+9C(q, \bar{h}_{20}, h_{21}) + 3C(q, q, \bar{h}_{31}) + 9C(q, \bar{q}, h_{22}) + 18C(\bar{q}, h_{11}, h_{21})$$
$$+9C(\bar{q}, h_{20}, \bar{h}_{21}) + 3C(\bar{q}, \bar{h}_{20}, h_{30}) + 3C(\bar{q}, \bar{q}, h_{31}) + 9D(q, q, \bar{h}_{20}, h_{11}) \quad (33)$$
$$+D(q, q, q, \bar{h}_{30}) + 9D(q, q, \bar{q}, \bar{h}_{21}) + 18D(q, \bar{q}, h_{11}, h_{11})$$
$$+9D(q, \bar{q}, \bar{h}_{20}, h_{20}) + 9D(q, \bar{q}, \bar{q}, h_{21}) + 9D(\bar{q}, \bar{q}, h_{11}, h_{20})$$
$$+3E(q, q, q, \bar{q}, \bar{h}_{20}) + 9E(q, q, \bar{q}, \bar{q}, h_{11}) + 3E(q, \bar{q}, \bar{q}, \bar{q}, h_{20})$$
$$+K(q, q, q, \bar{q}, \bar{q}, \bar{q}) - 3(G_{32} + \bar{G}_{32})h_{11} - 9(G_{21} + \bar{G}_{21})h_{22}].$$

Defining $\mathcal{H}_{43}$ as

$$\mathcal{H}_{43} = 12B(h_{11}, h_{32}) + 6B(h_{20}, \bar{h}_{32}) + 3B(\bar{h}_{20}, h_{41})$$

$$+18B(h_{21}, h_{22}) + 12B(\bar{h}_{21}, h_{31}) + 4B(h_{30}, \bar{h}_{31}) + B(\bar{h}_{30}, h_{40})$$
$$+4B(q, h_{33}) + 3B(\bar{q}, h_{42}) + 36C(h_{11}, h_{11}, h_{21}) + 36C(h_{11}, h_{20}, \bar{h}_{21})$$
$$+12C(h_{11}, \bar{h}_{20}, h_{30}) + 3C(h_{20}, h_{20}, \bar{h}_{30}) + 18C(h_{20}, \bar{h}_{20}, h_{21})$$
$$+36C(q, h_{11}, h_{22}) + 12C(q, h_{20}, \bar{h}_{31}) + 12C(q, \bar{h}_{20}, h_{31})$$
$$+36C(q, h_{21}, \bar{h}_{21}) + 4C(q, h_{30}, \bar{h}_{30}) + 6C(q, q, \bar{h}_{32})$$
$$+12C(q, \bar{q}, h_{32}) + 24C(\bar{q}, h_{11}, h_{31}) + 18C(\bar{q}, h_{20}, h_{22})$$
$$+3C(\bar{q}, \bar{h}_{20}, h_{40}) + 18C(\bar{q}, h_{21}, h_{21}) + 12C(\bar{q}, \bar{h}_{21}, h_{30})$$
$$+3C(\bar{q}, \bar{q}, h_{41}) + 24D(q, h_{11}, h_{11}, h_{11}) + 36D(q, h_{11}, h_{20}, \bar{h}_{20})$$
$$+36D(q, q, h_{11}, \bar{h}_{21}) + 6D(q, q, h_{20}, \bar{h}_{30}) + 18D(q, q, \bar{h}_{20}, h_{21})$$
$$+4D(q, q, q, \bar{h}_{31}) + 18D(q, q, \bar{q}, h_{22}) + 72D(q, \bar{q}, h_{11}, h_{21})$$
$$+36D(q, \bar{q}, h_{20}, \bar{h}_{21}) + 12D(q, \bar{q}, \bar{h}_{20}, h_{30}) + 12D(q, \bar{q}, \bar{q}, h_{31})$$
$$+36D(\bar{q}, h_{11}, h_{11}, h_{20}) + 9D(\bar{q}, h_{20}, h_{20}, \bar{h}_{20}) + 12D(\bar{q}, \bar{q}, h_{11}, h_{30})$$
$$+18D(\bar{q}, \bar{q}, h_{20}, h_{21}) + D(\bar{q}, \bar{q}, \bar{q}, h_{40}) + 12E(q, q, q, h_{11}, \bar{h}_{20})$$
$$+E(q, q, q, q, \bar{h}_{30}) + 12E(q, q, q, \bar{q}, \bar{h}_{21}) + 36E(q, q, \bar{q}, h_{11}, h_{11})$$
$$+18E(q, q, \bar{q}, h_{20}, \bar{h}_{20}) + 18E(q, q, \bar{q}, \bar{q}, h_{21}) + 36E(q, \bar{q}, \bar{q}, h_{11}, h_{20})$$
$$+4E(q, \bar{q}, \bar{q}, \bar{q}, h_{30}) + 3E(\bar{q}, \bar{q}, \bar{q}, h_{20}, h_{20}) + 3K(q, q, q, q, \bar{q}, \bar{h}_{20})$$
$$+12K(q, q, q, \bar{q}, \bar{q}, h_{11}) + 6K(q, q, \bar{q}, \bar{q}, \bar{q}, h_{20}) + L(q, q, q, q, \bar{q}, \bar{q}, \bar{q})$$
$$-6(2G_{32}h_{21} + \bar{G}_{32}h_{21} + 3G_{21}h_{32} + 2\bar{G}_{21}h_{32}),$$

and from the coefficients of the terms $w^4\bar{w}^3$ in (14), one has a singular system for $h_{43}$

$$(i\omega_0 I_n - A)h_{43} = \mathcal{H}_{43} - G_{43}q \tag{34}$$

which has solution if and only if

$$\langle p, \mathcal{H}_{43} - G_{43}q \rangle = 0. \tag{35}$$

where the terms $-6(2G_{32}h_{21} + \bar{G}_{32}h_{21} + 3G_{21}h_{32} + 2\bar{G}_{21}h_{32})$ appearing in the last line of equation (34) actually do not enter in the last equation, since $\langle p, h_{21} \rangle = 0$ and $\langle p, h_{32} \rangle = 0$.

The *third Lyapunov coefficient* is defined by

$$l_3 = \frac{1}{144} \operatorname{Re} G_{43}, \tag{36}$$

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where, from (35), $G_{43} = \langle p, \mathcal{H}_{43} \rangle$.

The complex vector $h_{43}$ can be found solving the nonsingular $(n + 1)$-dimensional system

$$\begin{pmatrix} i\omega_0 I_n - A & q \\ \\ \bar{p} & 0 \end{pmatrix} \begin{pmatrix} h_{43} \\ \\ s \end{pmatrix} = \begin{pmatrix} \mathcal{H}_{43} - G_{43}q \\ \\ 0 \end{pmatrix}, \tag{37}$$

with the condition $\langle p, h_{43} \rangle = 0$.

Defining $\mathcal{H}_{54}$ by the expression below

$$20B(h_{11}, h_{43}) + 10B(h_{20}, \bar{h}_{43}) + 6B(\bar{h}_{20}, h_{52}) + 40B(h_{21}, h_{33}) + 30B(\bar{h}_{21}, h_{42}) +$$
$$60B(h_{22}, h_{32}) + 10B(h_{30}, \bar{h}_{42}) + 4B(\bar{h}_{30}, h_{51}) + 40B(h_{31}, \bar{h}_{32}) + 20B(\bar{h}_{31}, h_{41}) +$$
$$5B(h_{40}, \bar{h}_{41}) + B(\bar{h}_{40}, h_{50}) + 5B(q, h_{44}) + 4B(\bar{q}, h_{53}) + 120C(h_{11}, h_{11}, h_{32}) +$$
$$60C(h_{11}, \bar{h}_{20}, h_{41}) + 360C(h_{11}, h_{21}, h_{22}) + 240C(h_{11}, \bar{h}_{21}, h_{31}) + 80C(h_{11}, h_{30}, \bar{h}_{31}) +$$
$$20C(h_{11}, \bar{h}_{30}, h_{40}) + 120C(h_{20}, h_{11}, \bar{h}_{32}) + 15C(h_{20}, h_{20}, \bar{h}_{41}) + 60C(h_{20}, \bar{h}_{20}, h_{32}) +$$
$$120C(h_{20}, h_{21}, \bar{h}_{31}) + 180C(h_{20}, \bar{h}_{21}, h_{22}) + 10C(h_{20}, h_{30}, \bar{h}_{40}) + 40C(h_{20}, \bar{h}_{30}, h_{31}) +$$
$$3C(\bar{h}_{20}, \bar{h}_{20}, h_{50}) + 120C(\bar{h}_{20}, h_{21}, h_{31}) + 30C(\bar{h}_{20}, \bar{h}_{21}, h_{40}) + 60C(\bar{h}_{20}, h_{30}, h_{22}) +$$
$$180C(h_{21}, h_{21}, \bar{h}_{21}) + 60C(\bar{h}_{21}, \bar{h}_{21}, h_{30}) + 40C(h_{30}, h_{21}, \bar{h}_{30}) + 80C(q, h_{11}, h_{33}) +$$
$$30C(q, h_{20}, \bar{h}_{42}) + 30C(q, \bar{h}_{20}, h_{42}) + 120C(q, h_{21}, \bar{h}_{32}) + 120C(q, \bar{h}_{21}, h_{32}) +$$
$$90C(q, h_{22}, h_{22}) + 20C(q, h_{30}, \bar{h}_{41}) + 20C(q, \bar{h}_{30}, h_{41}) + 80C(q, h_{31}, \bar{h}_{31}) + 5C(q, h_{40}, \bar{h}_{40}) +$$
$$10C(q, q, \bar{h}_{43}) + 20C(q, \bar{q}, h_{43}) + 60C(\bar{q}, h_{11}, h_{42}) + 40C(\bar{q}, h_{20}, h_{33}) + 12C(\bar{q}, \bar{h}_{20}, h_{51}) +$$
$$120C(\bar{q}, h_{21}, h_{32}) + 60C(\bar{q}, \bar{h}_{21}, h_{41}) + 40C(\bar{q}, h_{30}, \bar{h}_{32}) + 4C(\bar{q}, \bar{h}_{30}, h_{50}) +$$
$$120C(\bar{q}, h_{31}, h_{22}) + 20C(\bar{q}, h_{40}, \bar{h}_{31}) + 6C(\bar{q}, \bar{q}, h_{52}) + 240D(h_{11}, h_{11}, h_{11}, h_{21}) +$$
$$120D(h_{11}, h_{11}, \bar{h}_{20}, h_{30}) + 360D(h_{20}, h_{11}, h_{11}, \bar{h}_{21}) + 360D(h_{20}, h_{11}, \bar{h}_{20}, h_{21}) +$$
$$60D(h_{20}, h_{20}, h_{11}, \bar{h}_{30}) + 90D(h_{20}, h_{20}, \bar{h}_{20}, \bar{h}_{21}) + 30D(h_{20}, \bar{h}_{20}, \bar{h}_{20}, h_{30}) +$$
$$360D(q, h_{11}, h_{11}, h_{22}) + 240D(q, h_{11}, \bar{h}_{20}, h_{31}) + 720D(q, h_{11}, h_{21}, \bar{h}_{21}) +$$
$$80D(q, h_{11}, h_{30}, \bar{h}_{30}) + 240D(q, h_{20}, h_{11}, \bar{h}_{31}) + 15D(q, h_{20}, h_{20}, \bar{h}_{40}) +$$
$$180D(q, h_{20}, \bar{h}_{20}, h_{22}) + 120D(q, h_{20}, h_{21}, \bar{h}_{30}) + 180D(q, h_{20}, \bar{h}_{21}, \bar{h}_{21}) +$$
$$15D(q, \bar{h}_{20}, \bar{h}_{20}, h_{40}) + 180D(q, \bar{h}_{20}, \bar{h}_{20}, h_{21}) + 120D(q, \bar{h}_{20}, h_{30}, \bar{h}_{21}) +$$
$$120D(q, q, h_{11}, \bar{h}_{32}) + 30D(q, q, h_{20}, \bar{h}_{41}) + 60D(q, q, \bar{h}_{20}, h_{32}) + 120D(q, q, h_{21}, \bar{h}_{31}) +$$
$$180D(q, q, \bar{h}_{21}, h_{22}) + 10D(q, q, h_{30}, \bar{h}_{40}) + 40D(q, q, \bar{h}_{30}, h_{31}) + 10D(q, q, q, \bar{h}_{42}) +$$
$$40D(q, q, \bar{q}, h_{33}) + 240D(q, \bar{q}, h_{11}, h_{32}) + 120D(q, \bar{q}, h_{20}, \bar{h}_{32}) + 60D(q, \bar{q}, \bar{h}_{20}, h_{41}) +$$
$$360D(q, \bar{q}, h_{21}, h_{22}) + 240D(q, \bar{q}, \bar{h}_{21}, h_{31}) + 80D(q, \bar{q}, h_{30}, \bar{h}_{31}) + 20D(q, \bar{q}, \bar{h}_{30}, h_{40}) +$$
$$30D(q, \bar{q}, \bar{q}, h_{42}) + 240D(\bar{q}, h_{11}, h_{11}, h_{31}) + 60D(\bar{q}, h_{11}, \bar{h}_{20}, h_{40}) + 360D(\bar{q}, h_{11}, h_{21}, h_{21}) +$$
$$240D(\bar{q}, h_{11}, h_{30}, \bar{h}_{21}) + 360D(\bar{q}, h_{20}, h_{11}, h_{22}) + 60D(\bar{q}, h_{20}, h_{20}, \bar{h}_{31}) +$$
$$120D(\bar{q}, h_{20}, \bar{h}_{20}, h_{31}) + 360D(\bar{q}, h_{20}, h_{21}, \bar{h}_{21}) + 40D(\bar{q}, h_{20}, h_{30}, \bar{h}_{30}) +$$
$$120D(\bar{q}, \bar{h}_{20}, h_{30}, h_{21}) + 60D(\bar{q}, \bar{q}, h_{11}, h_{41}) + 60D(\bar{q}, \bar{q}, h_{20}, h_{32}) + 6D(\bar{q}, \bar{q}, h_{20}, h_{50}) +$$

$$120D(\bar{q}, \bar{q}, h_{21}, h_{31}) + 30D(\bar{q}, \bar{q}, \bar{h}_1, h_{40}) + 60D(\bar{q}, \bar{q}, h_{30}, h_{22}) + 4D(\bar{q}, \bar{q}, \bar{q}, h_{51}) +$$
$$120E(q, h_{11}, h_{11}, h_{11}, h_{11}) + 360E(q, h_{20}, h_{11}, h_{11}, \bar{h}_{20}) + 45E(q, h_{20}, h_{20}, \bar{h}_{20}, \bar{h}_{20}) +$$
$$360E(q, q, h_{11}, h_{11}, \bar{h}_{21}) + 360E(q, q, h_{11}, \bar{h}_{20}, h_{21}) + 120E(q, q, h_{20}, h_{11}, \bar{h}_{30}) +$$
$$180E(q, q, h_{20}, \bar{h}_{20}, \bar{h}_{21}) + 30E(q, q, \bar{h}_{20}, \bar{h}_{20}, h_{30}) + 80E(q, q, q, h_{11}, \bar{h}_{31}) +$$
$$10E(q, q, q, h_{20}, \bar{h}_{40}) + 60E(q, q, q, \bar{h}_{20}, h_{22}) + 40E(q, q, q, h_{21}, \bar{h}_{30}) + 60E(q, q, q, \bar{h}_{21}, \bar{h}_{21}) +$$
$$5E(q, q, q, q, \bar{h}_{41}) + 40E(q, q, q, \bar{q}, \bar{h}_{32}) + 360E(q, q, \bar{q}, h_{11}, h_{22}) + 120E(q, q, \bar{q}, h_{20}, \bar{h}_{31}) +$$
$$120E(q, q, \bar{q}, \bar{h}_{20}, h_{31}) + 360E(q, q, \bar{q}, h_{21}, \bar{h}_{21}) + 40E(q, q, \bar{q}, h_{30}, \bar{h}_{30}) + 60E(q, q, \bar{q}, \bar{q}, \bar{h}_{32}) +$$
$$720E(q, \bar{q}, h_{11}, h_{11}, h_{21}) + 240E(q, \bar{q}, h_{11}, \bar{h}_{20}, h_{30}) + 720E(q, \bar{q}, h_{20}, h_{11}, \bar{h}_{21}) +$$
$$60E(q, \bar{q}, h_{20}, h_{20}, \bar{h}_{30}) + 360E(q, \bar{q}, h_{20}, \bar{h}_{20}, h_{21}) + 240E(q, \bar{q}, \bar{q}, h_{11}, h_{31}) +$$
$$180E(q, \bar{q}, \bar{q}, h_{20}, h_{22}) + 30E(q, \bar{q}, \bar{q}, \bar{h}_{20}, h_{40}) + 180E(q, \bar{q}, \bar{q}, h_{21}, h_{21}) + 120E(q, \bar{q}, \bar{q}, h_{30}, \bar{h}_{21}) +$$
$$20E(q, \bar{q}, \bar{q}, \bar{q}, h_{41}) + 240E(\bar{q}, h_{20}, h_{11}, h_{11}, h_{11}) + 180E(\bar{q}, h_{20}, h_{20}, h_{11}, \bar{h}_{20}) +$$
$$120E(\bar{q}, \bar{q}, h_{11}, h_{11}, h_{30}) + 360E(\bar{q}, \bar{q}, h_{20}, h_{11}, h_{21}) + 90E(\bar{q}, \bar{q}, h_{20}, h_{20}, \bar{h}_{21}) +$$
$$60E(\bar{q}, \bar{q}, h_{20}, \bar{h}_{20}, h_{30}) + 20E(\bar{q}, \bar{q}, \bar{q}, h_{11}, h_{40}) + 40E(\bar{q}, \bar{q}, \bar{q}, h_{20}, h_{31}) + 40E(\bar{q}, \bar{q}, \bar{q}, h_{30}, h_{21}) +$$
$$E(\bar{q}, \bar{q}, \bar{q}, \bar{q}, h_{50}) + 120K(q, q, q, h_{11}, h_{11}, \bar{h}_{20}) + 30K(q, q, q, h_{20}, \bar{h}_{20}, \bar{h}_{20}) +$$
$$20K(q, q, q, q, h_{11}, \bar{h}_{30}) + 30K(q, q, q, q, \bar{h}_{20}, \bar{h}_{21}) + K(q, q, q, q, q, \bar{h}_{40}) + 20K(q, q, q, q, \bar{q}, \bar{h}_{31}) +$$
$$240K(q, q, q, \bar{q}, h_{11}, \bar{h}_{21}) + 40K(q, q, q, \bar{q}, h_{20}, \bar{h}_{30}) + 120K(q, q, q, \bar{q}, \bar{h}_{20}, h_{21}) +$$
$$60K(q, q, q, \bar{q}, \bar{q}, h_{22}) + 240K(q, q, \bar{q}, h_{11}, h_{11}, h_{11}) + 360K(q, q, \bar{q}, h_{20}, h_{11}, \bar{h}_{20}) +$$
$$360K(q, q, \bar{q}, \bar{q}, h_{11}, h_{21}) + 180K(q, q, \bar{q}, \bar{q}, h_{20}, \bar{h}_{21}) + 60K(q, q, \bar{q}, \bar{q}, \bar{h}_{20}, h_{30}) +$$
$$40K(q, q, \bar{q}, \bar{q}, \bar{q}, h_{31}) + 360K(q, \bar{q}, \bar{q}, h_{20}, h_{11}, h_{11}) + 90K(q, \bar{q}, \bar{q}, h_{20}, h_{20}, \bar{h}_{20}) +$$
$$80K(q, \bar{q}, \bar{q}, \bar{q}, h_{11}, h_{30}) + 120K(q, \bar{q}, \bar{q}, \bar{q}, h_{20}, h_{21}) + 5K(q, \bar{q}, \bar{q}, \bar{q}, \bar{q}, h_{40}) +$$
$$60K(\bar{q}, \bar{q}, \bar{q}, h_{20}, h_{20}, h_{11}) + 10K(\bar{q}, \bar{q}, \bar{q}, \bar{q}, h_{20}, h_{30}) + 3L(q, q, q, q, q, \bar{h}_{20}, \bar{h}_{20}) +$$
$$4L(q, q, q, q, q, \bar{q}, \bar{h}_{30}) + 60L(q, q, q, q, \bar{q}, h_{11}, \bar{h}_{20}) + 30L(q, q, q, q, \bar{q}, \bar{q}, \bar{h}_{21}) +$$
$$120L(q, q, q, \bar{q}, \bar{q}, h_{11}, h_{11}) + 60L(q, q, q, \bar{q}, \bar{q}, h_{20}, \bar{h}_{20}) + 40L(q, q, q, \bar{q}, \bar{q}, \bar{q}, h_{21}) +$$
$$120L(q, q, \bar{q}, \bar{q}, \bar{q}, h_{20}, h_{11}) + 10L(q, q, \bar{q}, \bar{q}, \bar{q}, \bar{q}, h_{30}) + 15L(q, \bar{q}, \bar{q}, \bar{q}, \bar{q}, h_{20}, h_{20}) +$$
$$6M(q, q, q, q, q, \bar{q}, \bar{q}, \bar{h}_{20}) + 20M(q, q, q, q, \bar{q}, \bar{q}, \bar{q}, h_{11}) + 10M(q, q, q, \bar{q}, \bar{q}, \bar{q}, \bar{q}, h_{20}) +$$
$$N(q, q, q, q, q, \bar{q}, \bar{q}, \bar{q}, \bar{q}),$$

and from the coefficients of the terms $w^5 \bar{w}^4$ in (14), one has a singular system for $h_{54}$

$$(i\omega_0 I_n - A)h_{54} = \mathcal{H}_{54} - G_{54}q \tag{38}$$

which has solution if and only if

$$\langle p, \mathcal{H}_{54} - G_{54}q \rangle = 0. \tag{39}$$

The *fourth Lyapunov coefficient* is defined by

$$l_4 = \frac{1}{2880} \operatorname{Re} G_{54}, \tag{40}$$

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where, from (39), $G_{54} = \langle p, \mathcal{H}_{54} \rangle$.

**Remark 1.2** *Other equivalent definitions and algorithmic procedures to write the expressions for the Lyapunov coefficients $l_j, j = 1, 2, 3, 4$, for two dimensional systems can be found in Andronov et al. [1] and Gasull et al. [2], among others. These procedures apply also to the $n$–dimensional systems of this work, if properly restricted to the center manifold. The authors found, however, that the method outlined above, due to Kuznetsov [3, 4], requiring no explicit formal evaluation of the center manifold, is better adapted to the needs of long calculations in [5, 6, 7], and for that in [8], where $n = 3$.*

A *Hopf point* $(\mathbf{x_0}, \mu_0)$ is an equilibrium point of (1) where the Jacobian matrix $A = f_{\mathbf{x}}(\mathbf{x_0}, \mu_0)$ has a pair of purely imaginary eigenvalues $\lambda_{2,3} = \pm i\omega_0$, $\omega_0 > 0$, and admits no other critical eigenvalues —i.e. located on the imaginary axis. At a Hopf point a two dimensional center manifold is well-defined, it is invariant under the flow generated by (1) and can be continued with arbitrary high class of differentiability to nearby parameter values. In fact, what is well defined is the $\infty$-jet —or infinite Taylor series— of the center manifold, as well as that of its continuation, any two of them having contact in the arbitrary high order of their differentiability class.

A Hopf point is called *transversal* if the parameter dependent complex eigenvalues cross the imaginary axis with non-zero derivative. In a neighborhood of a transversal Hopf point —H1 point, for concision— with $l_1 \neq 0$ the dynamic behavior of the system (1), reduced to the family of parameter-dependent continuations of the center manifold, is orbitally topologically equivalent to the following complex normal form

$$w' = (\eta + i\omega)w + l_1 w |w|^2,$$

$w \in \mathbb{C}$, $\eta$, $\omega$ and $l_1$ are real functions having derivatives of arbitrary high order, which are continuations of $0$, $\omega_0$ and the first Lyapunov coefficient at the H1 point. See [3]. As $l_1 < 0$ ($l_1 > 0$) one family of stable (unstable)

periodic orbits can be found on this family of manifolds, shrinking to an equilibrium point at the H1 point.

A *Hopf point of codimension 2* is a Hopf point where $l_1$ vanishes. It is called *transversal* if $\eta = 0$ and $l_1 = 0$ have transversal intersections, where $\eta = \eta(\mu)$ is the real part of the critical eigenvalues. In a neighborhood of a transversal Hopf point of codimension 2 —H2 point, for concision— with $l_2 \neq 0$ the dynamic behavior of the system (1), reduced to the family of parameter-dependent continuations of the center manifold, is orbitally topologically equivalent to

$$w' = (\eta + i\omega_0)w + \tau w|w|^2 + l_2 w|w|^4,$$

where $\eta$ and $\tau$ are unfolding parameters. See [3]. The bifurcation diagrams for $l_2 \neq 0$ can be found in [3], p. 313, and in [9].

A *Hopf point of codimension 3* is a Hopf point of codimension 2 where $l_2$ vanishes. A Hopf point of codimension 3 point is called *transversal* if $\eta = 0$, $l_1 = 0$ and $l_2 = 0$ have transversal intersections. In a neighborhood of a transversal Hopf point of codimension 3 —H3 point, for concision— with $l_3 \neq 0$ the dynamic behavior of the system (1), reduced to the family of parameter-dependent continuations of the center manifold, is orbitally topologically equivalent to

$$w' = (\eta + i\omega_0)w + \tau w|w|^2 + \nu w|w|^4 + l_3 w|w|^6,$$

where $\eta$, $\tau$ and $\nu$ are unfolding parameters. The bifurcation diagram for $l_3 \neq 0$ can be found in Takens [9] and in [7].

A *Hopf point of codimension 4* is a Hopf point of codimension 3 where $l_3$ vanishes. A Hopf point of codimension 4 is called *transversal* if $\eta = 0$, $l_1 = 0$, $l_2 = 0$ and $l_3 = 0$ have transversal intersections. In a neighborhood of a transversal Hopf point of codimension 4 —H4 point, for concision— with $l_4 \neq 0$ the dynamic behavior of the system (1), reduced to the family of parameter-dependent continuations of the center manifold, is orbitally

topologically equivalent to

$$w' = (\eta + i\omega_0)w + \tau w|w|^2 + \nu w|w|^4 + \sigma w|w|^6 + l_4 w|w|^8,$$

where $\eta$, $\tau$, $\nu$ and $\sigma$ are unfolding parameters.

**Theorem 1.3** *Suppose that the system*

$$\mathbf{x}' = f(\mathbf{x}, \mu), \ \mathbf{x} = (x, y, z), \ \mu = (\beta, \alpha, \kappa, \varepsilon)$$

*has the equilibrium $\mathbf{x} = \mathbf{0}$ for $\mu = 0$ with eigenvalues*

$$\lambda_{2,3}(\mu) = \eta(\mu) \pm i\omega(\mu),$$

*where $\omega(0) = \omega_0 > 0$. For $\mu = 0$ the following conditions hold*

$$\eta(0) = 0, \ l_1(0) = 0, \ l_2(0) = 0, \ l_3(0) = 0,$$

*where $l_1(\mu)$, $l_2(\mu)$ and $l_3(\mu)$ are the first, second and third Lyapunov coefficients, respectively. Assume that the following genericity conditions are satisfied*

1. *$l_4(0) \neq 0$, where $l_4(0)$ is the fourth Lyapunov coefficient;*

2. *the map $\mu \to (\eta(\mu), l_1(\mu), l_2(\mu), l_3(\mu))$ is regular at $\mu = 0$.*

*Then, by the introduction of a complex variable, the above system reduced to the family of parameter-dependent continuations of the center manifold, is orbitally topologically equivalent to*

$$w' = (\eta + i\omega_0)w + \tau w|w|^2 + \nu w|w|^4 + \sigma w|w|^6 + l_4 w|w|^8$$

*where $\eta$, $\tau$, $\nu$ and $\sigma$ are unfolding parameters.*

**Remark 1.4** *The expressions for the Lyapunov coefficients in this article will be of interest in the study of the Hopf bifurcations of codimensions 1, 2, 3 and 4 in the Watt governor system with a spring [8], pursuing previous bifurcation analysis carried out in [7].*

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